FRICTION AND HEAT TRANSFER IN THE GAP BETWEEN TWO ROTATING COAXIAL CYLINDERS

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An attempt is made at a theoretical analysis of the friction and heat transfer in the gap between two coaxial cylinders rotating in one direction. The interaction of a fluid with the channel walls is divided into two stages: 1) formation of roller, annular fluid flows in the gap; and 2) friction and heat transfer with forced flow past the walls.

A great deal of experimental and theoretical work has been devoted to friction and heat-transfer processes in the gap between two rotating coaxial cylinders, but so far there is no theory that determines friction and heat transfer in the gap when roller, annular flows are formed.

A method for calculating friction at the moment of stability loss is proposed in [1]. Below, we discuss an analytic method for calculating friction and heat transfer in the gap between coaxial cylinders rotating in one direction when $l/r_{av} \ll 1$, $+\omega_1 > \mp\omega_2$, and axial flow is absent. In addition, unlike [1], we consider the domain of considerable values $Ta \gg Ta_{cr}$, in which the assumptions of the small-perturbation method, based on linearization of the equations of motion and energy, are inapplicable.

It is well known that at Taylor numbers greater than 1700, flow in the gap between two coaxial cylinders has a clearly expressed roller structure [2, 3]. Such a structure is preserved up to $Ta \approx 5 \cdot 10^5$ [3]. A diagram of fluid flow under the given conditions is shown in Fig. 1a and c.

We assume that friction and heat transfer under the conditions in question are chiefly dependent on the circulation velocity of the fluid in the annular rollers. In this case, the problem can be reduced to a calculation of the dynamic and thermal boundary layers produced on the surface of the body as a result of the interaction of paired rollers, which form a cellular flow structure in the gap. The circulation velocity in the rollers is determined by the difference between the centrifugal forces in the rising and descending flows through the boundary layers on the surfaces of the cylinder. The centrifugal forces do not exert an appreciable influence on the dynamic and thermal boundary layers at the wall. Thus, the problem of friction and heat transfer under the conditions in question can be divided into two stages:

1) determination of the intensity of fluid circulation in the annular rollers; and

2) calculation of the dynamic and thermal boundary layers for forced flow near the surfaces of the cylinders.

Let us consider the flow of a fluid with constant physical parameters in a region of stable flow structure. Following Batchelor [4], we assume that at high Taylor numbers, flow in the cell core is characterized by a constant vorticity $\omega = \text{const}$. Then the solution of the equation $\nabla^2\psi = \omega$ when $\psi = 0$ at the roller boundaries has the form

$$\psi = \sum_{n=1}^{n=\infty} \frac{2\omega(1-\cos n\pi)}{l\lambda^3}\left[\left(\frac{1}{\operatorname{sh}\lambda l} - \operatorname{cth}\lambda l\right)\times \right.$$

$$\left. \times \operatorname{sh}\lambda y + \operatorname{ch}\lambda y - 1\right]\sin\lambda x, \quad (1)$$

where $\lambda = n\pi/l$. Hence, the horizontal velocity component is

$$u = \sum_{n=1}^{n=\infty} \frac{2\omega(1-\cos n\pi)}{l\lambda^2}\times$$

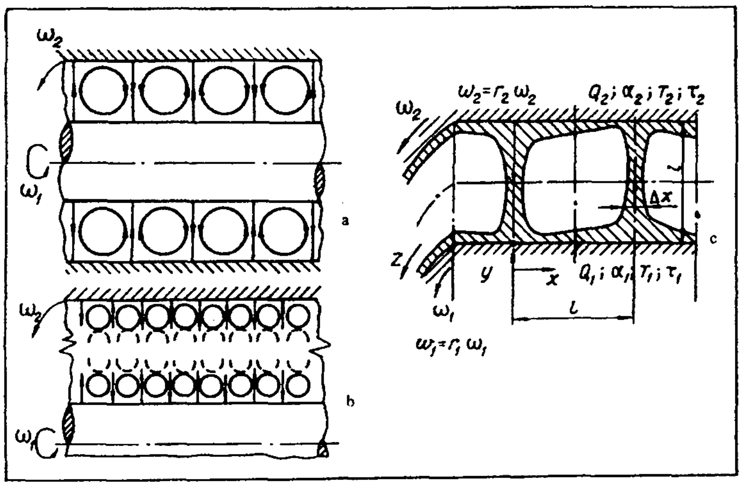

Fig. 1. Fluid flow in a coaxial gap.

$$\times\left[\left(\frac{1}{\operatorname{sh}\lambda l}-\operatorname{cth}\lambda l\right)\operatorname{ch}\lambda y+\operatorname{sh}\lambda y\right]\sin\lambda x. \quad (2)$$

Limiting ourselves to the first term of the series, for $y = 0$ we have

$$u_0 = v_0 \sin\frac{\pi x}{l}. \quad (3)$$

Equation (3) contains the so-far unknown maximum velocity v_0 in the roller. We have the following system of equations for determining this velocity and the friction and heat-transfer coefficients:

1. The momentum equation for the boundary layer that grows on the surface from forced fluid flow with velocity u_0 is, from [5],

$$\frac{d\,\mathrm{Re}^{**}}{dX}+\left(1+\frac{\delta^*}{\delta^{**}}\right)\frac{\mathrm{Re}^{**}}{u_0}\frac{du_0}{dX}=\mathrm{Re}_l\frac{c_f}{2}, \quad (4)$$

$$(\mathrm{Re}^{**})_{x=0}=0.$$

2. The friction law for a laminar boundary layer [5] is

$$\frac{c_f}{2}=\frac{0.22}{\mathrm{Re}^{**}}. \quad (5)$$

3. The dynamic boundary-layer equation for rotation of cylinders with a Z component is

$$u\frac{\partial\overline{w}}{\partial x}+v\frac{\partial\overline{w}}{\partial y}=\nu\frac{\partial^2\overline{w}}{\partial y^2}, \quad (6)$$

where $w = (w - w_0)/(w_1 - w_0)$, $\overline{w} = 1$ when $y = 0$, $\overline{w} = 0$ when $y = \infty$, and $\overline{w} = 0$ when $x = 0$.

4. The continuity equation is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \quad (7)$$

5. The angular-momentum equation for the roller volume is

$$\int_V(\mathbf{R}\cdot\rho\mathbf{F})\,dV+\int_\sigma(\mathbf{R}\cdot\mathbf{P}_n)\,d\sigma=0. \quad (8)$$

6. The thermal boundary-layer equation for Δt = const [5] is

$$\frac{d\,\mathrm{Re}^{**}_{\mathrm{inst}}}{dX}=\mathrm{Re}_l\,\mathrm{St}. \quad (9)$$

7. The heat-transfer law for a laminar boundary layer [5,6] is

$$\mathrm{St}=\frac{0.22}{\mathrm{Re}^{**}_{\mathrm{inst}}\,\mathrm{Pr}^{1.2}}. \quad (10)$$

In addition, we have the condition that the friction torques on the inside and outside cylinders be equal. From Eqs. (4) and (5), under the initial condition $(\mathrm{Re}^{**})_{X=0} = 0$ and $\delta^*/\delta^{**} = 2.5$, we have

$$\mathrm{Re}^{**}=0.663\left(\frac{lv_0}{\nu\pi}f(X)\right)^{1/2}, \quad (11)$$

where

$$f(X)=(\sin\pi X)^{-7}\Big[0.546\left(\frac{\pi X}{2}-\frac{1}{4}\sin 2\pi X\right)-$$
$$-0.125(\sin\pi X)^7\cos\pi X-$$
$$-0.146(\sin\pi X)^5\cos\pi X-0.182(\sin\pi X)^3\cos\pi X\Big]. \quad (12)$$

If we substitute Re^{**} of (11) into (5), we obtain the following expression for the friction coefficient:

$$\tau_w=z(X)\rho v_0^{3/2}\left(\frac{\nu}{l}\right)^{1/2}. \quad (13)$$

The values of function Z(X) are: 0, 0.565, 1.110, 1.230, 1.140, 0.895, 0.537, 0.201, 0.184, 0.010, and 0 for X = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, and 1, respectively.

For the mean tangential stress on the wall from one roller, we have the formula

$$\overline{\tau}_w=0.574\,\rho v_0^{3/2}\left(\frac{\nu}{l}\right)^{1/2}. \quad (14)$$

In solving Eq. (6) we use an asymptotic distribution of the velocities near the wall [6]: $u = \tau_w y/\mu$. In this

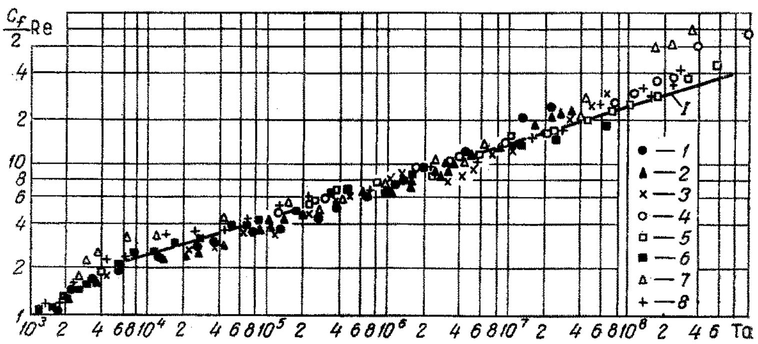

Fig. 2. Comparison of experimental data on friction coefficients with calculation results (water, water-glycerin mixture, heptane): 1) l/r_{av} = 0.0275; 2) 0.0403; 3) 0.0560; 4) 0.16; 5) 0.234 (1–5 from [10]); 6) l/r_{av} = 0.066; 7) 0.38; 8) 0.162; (6–8 from [11]); I) by formula (32).

case, Eq. (6) can be represented as

$$\frac{\partial^2 \bar{w}}{\partial y^2} + A(x)\, y^2 \frac{\partial \bar{w}}{\partial y} = B(x)\, y \frac{\partial \bar{w}}{\partial x}, \qquad (15)$$

where

$$A(x) = \frac{\partial \tau_w}{\partial x} \frac{1}{2\nu\mu};\quad B(x) = \frac{\tau_w}{\nu\mu}.$$

Let us integrate Eq. (15) under the boundary conditions (6) using the substitution of M. E. Shvets [7]:

$$\eta = y \tau_w^{1/2} \left(\int_0^x \mu\nu\tau_w^{1/2} dx \right)^{-1/3}. \qquad (16)$$

Equation (15) can be transformed in the new variables to

$$\frac{d^2 \bar{w}}{d\eta^2} + \frac{\eta^2}{3} \frac{d\bar{w}}{d\eta} = 0. \qquad (17)$$

The solution of (17) that satisfies the boundary conditions $w = 1$ when $\eta = 0$ and $\bar{w} = 0$ as $\eta \to \infty$ has the form

$$\bar{w} = 1 - \frac{\int_0^\eta \exp\left[-\frac{\eta^3}{9}\right] d\eta}{\int_0^\infty \exp\left[-\frac{\eta^3}{9}\right] d\eta}, \qquad (18)$$

where

$$\int_0^\infty \exp\left[-\frac{\eta^3}{9}\right] d\eta = \frac{1}{3^{1/3}} \int_0^\infty e^{-\xi} \xi^{(1/3-1)} d\xi = \frac{\Gamma(1/3)}{3^{1/3}} = 1.855.$$

From (18) we obtain the local friction coefficient

$$\tau = \frac{1}{1.855} (w_0 - w_1) \frac{\mu}{(\mu\nu)^{1/3}} \tau_w^{1/2} \left[\int_0^x \tau_w^{1/2} dx \right]^{-1/3}. \qquad (19)$$

For the mean friction coefficient, considering (13) and (19), we have the formula

$$\bar{\tau} = 0.596\rho (w_0 - w_1) \left(\frac{\nu v_0}{l} \right)^{1/2}. \qquad (20)$$

To close the system of equations, it is necessary to determine the total centrifugal forces acting on the fluid circulating in the roller.

Integrating (6) with respect to y from 0 to δ and taking (7) into account, we have

$$\left[\int_0^\delta \rho u (w - w_0)\, dy \right]_0^x = -\int_0^x \tau\, dx, \qquad (21)$$

where the left side of the equation represents the variation of the flow of momentum in the roller from rotation of the cylinder.

From relations (20) and (21) it follows that

$$\left[\int_0^\delta \rho u (w - w_0)\, dy \right]_0^l = 0.596\rho (w_1 - w_0)(\nu v_0 l)^{1/2}. \qquad (22)$$

At high Ta, $\delta/l \ll 1$ and a change in rotation velocity occurs in the wall layer. The change in rotation velocity in circulating flows occurs in a thin layer Δx, which is commensurate with thickness δ.

In the limiting case of high Ta, we assume that in the region of rotation-velocity variation in circulating roller flows the velocity is determined from relation (2) at $x = 0$: $v = v_0 \sin(\pi/l)y$.

As follows from the adopted model, centrifugal forces act only on that part of the fluid that flows through the boundary layer, and in the region of rising and falling flows it is bounded by the flow lines of an ideal fluid in the core of the roller. Then, $\Delta x v = \text{const}$. We define v as an average over the width of the gap, wherein $\Delta x_{av} v_{av} = \text{const}$:

$$v_{av} = v_0 \frac{1}{l} \int_0^l \sin \frac{\pi}{l} y dy = 0.637 v_0. \qquad (23)$$

In this case, the difference between the centrifugal forces in roller flows that circulate over the width of the gap is determined by the relation

$$W_1 = \frac{\left[\int_0^\delta \rho u (w - w_0)\, dy \right]_0^l}{\rho v_{av} \Delta x_{av}} \frac{w_0 + w_0'}{r_{av}} \rho l \Delta x_{av}, \qquad (24)$$

where $\frac{\left[\int_0^\delta \rho u (w - w_0)\, dy \right]_0^l}{\rho \Delta x_{av} v_{av}} = \Delta w_{av}$ is the difference between the average rotation velocities in circulating

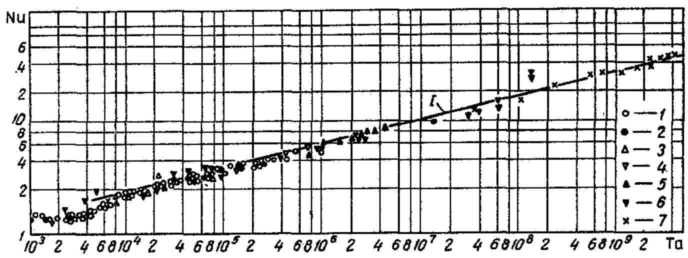

Fig. 3. Comparison of experimental data on heat-transfer coefficients with calculation results: 1) $l/r_{av} = 0.214$ [12]; 2) $l/r_{av} = 0.218$; 3) 0.054; 4) 0.084; 5) 0.246 (2–5 from [13]); 6) $l/r_{av} = 0.333$ [14]; 7) $l/r_{av} = 0.38$ [15]; I) by formula (37).

roller flows over the width of the layer. The friction force on the inside cylinder is

$$W' = \bar{\tau} l = 0.596\, l \rho (w_0 - w_1) \left(\frac{\nu v_0}{l}\right)^{1/2}. \tag{25}$$

The friction force on the outside cylinder is

$$W'' = 0.596\, l \rho \left[(w_0 + \Delta w_{av}) - w_2\right] \left(\frac{\nu v_0}{l}\right)^{1/2}. \tag{26}$$

When $l \ll r_{av}$, $|W'| = |W''|$; then

$$w_0 = \frac{w_1 + w_2 - \Delta w_{av}}{2}. \tag{27}$$

Since $\Delta w_{av} \ll w_1 + w_2$, in the first approximation, we take

$$w_0 = \frac{w_1 + w_2}{2}. \tag{28}$$

Considering (22), (23), (24), and (28), for $l/r \ll 1$ we have

$$W_1 = 0.469 \left(w_1^2 - w_2^2\right) \rho \frac{l}{r_{av}} \left(\frac{\nu l}{v_0}\right)^{1/2}. \tag{29}$$

As has already been indicated, at high Ta viscous forces appear in a thin layer about the periphery of the roller. Assuming R = const in the first approximation for a symmetric roller, we can transform Eq. (8) to

$$\int_V \rho F\, dV + \int_\sigma P_s\, d\sigma = W_1 + W. \tag{30}$$

If we equate the friction forces $\left(W = \int_\delta P_s\, d\sigma = 2\bar{\tau}_w l\right)$ and the difference between the centrifugal forces in the vortex flows (29), we obtain

$$v_0 = 0.64 \frac{\nu}{l} \mathrm{Ta}^{1/2}. \tag{31}$$

If we substitute v_0 into Eq. (20) and take (28) into account, we obtain the relation

$$\frac{c_f}{2} \mathrm{Re} = 0.238 \mathrm{Ta}^{1/4}, \tag{32}$$

where

$$\frac{c_f}{2} = \frac{\bar{\tau}}{(w_1 - w_2)^2 \rho}; \quad \mathrm{Re} = \frac{w_1 - w_2}{\nu} l.$$

From Eqs. (9) and (10) we have

$$\mathrm{Re}^{**}_{inst} = \left[\frac{0.44 v_0 l}{\pi \nu \mathrm{Pr}^{1.2}} \left(1 - \cos\frac{\pi x}{l}\right)\right]^{1/2}. \tag{33}$$

If we substitute (33) into (10), we obtain a formula for the mean heat-transfer coefficient

$$\bar{\alpha} = 0.53 \lambda \mathrm{Pr}^{0.4} \left(\frac{v_0}{l \nu}\right)^{1/2}. \tag{34}$$

On the basis of equality of the thermal fluxes on the inside and outside cylinders ($r_{av} \gg l$), we obtain

$$t_0 = \frac{t_1 + t_2 - \Delta t_{av}}{2}. \tag{35}$$

Since $\Delta t_{av} \ll t_1 + t_2$, in the first approximation we can take

$$t_0 = \frac{t_1 + t_2}{2}. \tag{36}$$

Substituting v_0 of (31) into relation (34) and assuming that t_0 is defined by formula (36), we obtain

$$\mathrm{Nu} = 0.212 \mathrm{Pr}^{0.4} \mathrm{Ta}^{1/4}. \tag{37}$$

In Figs. 2 and 3, the results of calculations by formulas (32) and (37) are compared with the available experimental data. As can be seen from the graphs, the theoretical calculations are in good agreement with the experimental data on the friction coefficient as well as with those for the heat-transfer coefficient.

Starting from the analogy of roller-flow processes in the gap between two coaxial cylinders that rotate in the same direction, when $l \ll r_{av}$, and in a horizontal slit that is heated from below [8, 9], we can assume that as Ta → ∞ there will be a vortex layer near the surfaces of the cylinders (Fig. 1b). The Ta number calculated from the height of the vortex layer is equal to the critical value of the stability parameter (Ta_{cr}). Then the height of the vortex layer is determined by the relation

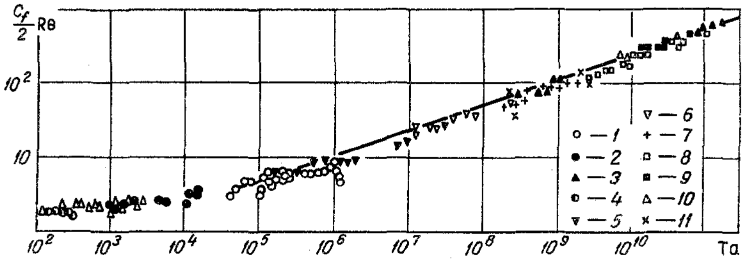

Fig. 4. Comparison of experimental data [16] on friction coefficient with calculation results: 1–4) various oils, r = 6.35 mm; 5) oil with kerosene, r = 9.5 mm; 6) kerosene, r = 6.35 mm; 7) water, r = 6.35 mm; 8) water, r = 12.7 mm; 9 and 10) air, r = 76 mm; 11) air, r = = 30 mm; I) by formula (41); $\mathrm{Ta} = w_1^2 r_1^3 / r_1 \nu^2$; $\mathrm{Re} = w_1 r_1 / \nu$.

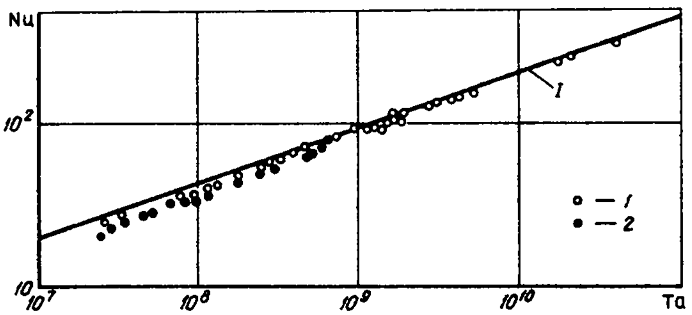

Fig. 5. Comparison of experimental data on heat-transfer coefficients with calculation results: 1) from [17]; 2) from [18]; I) by formula (42); $Ta = w_1^2 r_1^3/r_1\nu^2$.

$$l = \left[\frac{v^2 r_{av} \mathrm{Ta}_{cr}}{w_1^2 - w_0^2}\right]^{1/3}, \tag{38}$$

where w_0 is the mean rotation velocity over the cylinder surface. Assuming that in the central part of the gap the rotation velocity is constant over the cross section owing to strong turbulent mixing and assuming that $|\bar{\tau}_1| = |\bar{\tau}_2|$ when $l \ll r_{av}$, we have

$$w_0 = \frac{w_1 + w_2}{2}. \tag{39}$$

If we solve system (4)–(10), we find that v_0 is determined by relation (31).

Substituting relations (31), (38), and (39) into Eq. (20), we have

$$\frac{\bar{\tau}}{(w_1 - w_2)\rho v} = \frac{0.238}{(4\mathrm{Ta}_{cr})^{1/12}}\left(\frac{w_1^2 - w_2^2}{v^2 r_{av}}\right)^{1/4} \times \left[\frac{4w_1^2 - (w_1 + w_2)^2}{v^2 r_{av}}\right]^{1/12}. \tag{40}$$

When $w_2 = 0$ and $\mathrm{Ta}_{cr} = 1700$, we obtain

$$\frac{c_f}{2}\,\mathrm{Re} = \frac{0.238}{(4/3\,\mathrm{Ta}_{cr})^{1/12}}\,\mathrm{Ta}^{1/3} = 0.124\mathrm{Ta}^{1/3}, \tag{41}$$

where $\mathrm{Re} = w_1 r_1/\nu$.

The heat-transfer problem is solved similarly. Substituting v_0 into Eq. (34) and assuming that $\alpha \ll \alpha$ in the layer, when $w_2 = 0$ we obtain

$$\mathrm{Nu} = \frac{0.212}{\left(\frac{4}{3}\,\mathrm{Ta}_{cr}\right)^{1/12}}\,\mathrm{Pr}^{0.4}\mathrm{Ta}^{1/3} = 0.11\mathrm{Pr}^{0.4}\mathrm{Ta}^{1/3}. \tag{42}$$

It follows from relations (41) and (42) that the friction and heat-transfer coefficients for high Ta are not functions of the gap size.

Figures 4 and 5 show experimental data on the friction and heat-transfer coefficients on a smooth surface when the cylinder rotates in free space. This can be considered as the case of very high $\mathrm{Ta} \to \infty$. As can be seen from the graphs, the results of calculation by formulas (41) and (42) are in satisfactory agreement with the experimental data.

NOTATION

$w_1 = \omega_1 r_1$ is the rate of rotation of the internal cylinder; $w_2 = \omega_2 r_2$ is the rate of rotation of the external cylinder; $\mathrm{Ta} = (w_1^2 - w_2^2)l^3/r_{av}\nu^2$ is the Taylor number; $l = r_2 - r_1$ is the coaxial gap width; ν is the kinematic viscosity; v and u are, respectively, the radial and axial velocity components; $\mathrm{Re}^{**} = \mu_0\delta^{**}/\nu$ is the instantaneous value of the Reynolds number, plotted according to the momentum thickness; δ^{**} is the momentum thickness; δ^* is the displacement thickness; $\mathrm{Re}_l = u_0 l/\nu$ is the Reynolds number, plotted according to the instantaneous value of the velocity and the characteristic dimension; $X = x/l$ is the relative axial coordinate; c_f is the friction coefficient; St is the Stanton number; $\mathrm{Re}_{inst}^{**} = u_0\delta_{inst}^{**}/\nu$ is the instantaneous value of the Reynolds number, plotted according to the energy thickness; ρ is the density; γ is the specific weight; λ is the thermal-conductivity coefficient; Ta_{cr} is the Taylor number, plotted according to the height of the near-wall, roller cell layer; Pr is the Prandtl number; τ_w is the axial component of the friction coefficient for one roller; τ is the azimuthal component of the friction coefficient; t is the temperature; Nu is the Nusselt number; t_1 and t_2 are the temperatures of the internal and external cylinders, respectively; t_0 is the mean temperature in the flow of roller directed along the normal to the internal cylinder; F is the acceleration of mass forces; σ is the roller surface; P_s are surface forces; **R** is a vector directed along the shortest distance from the roller rotational axis to the considered point of the liquid and is equal in magnitude to this distance.

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